Solving Super-Size Problems with Optimization

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Very Large Scale Optimization Problems

» Schedule crews for 3,400 daily flights in 40 countries
» Buy ads in 10-15 local publications across 40,000 zip codes
» Pick one of 742 trillion choices in creating the NFL schedule
» Select 5 offers out of 1,000 for each of 25,000,000 customers
» Place 1,000s of SKUs on dozens of shelves in ~2,000 stores
» Decide among 200,000,000 maintenance routing options
» Plan weekly production levels for several years ahead

in pursuit of  
Conflicting business objectives (goals)

and subject to  
Multiple conflicting restrictions (constraints)

Must solve (close to) optimally, or your competition will
Benefits

» Regulatory and labor compliance while maximizing productivity

» Increased decision precision, saving millions of dollars per year

» Create schedules that satisfy all stakeholders

» Improve response rates from 1-2% to 5-10%

» Consistent and faster retail planning results in hundreds of thousands of dollars in both cost reduction and revenue increase per store

» Improve rental fleet utilization to save $19M

» Food manufacturing efficiency improvements of millions of dollars per production line
Agenda

Large-Scale Problem Types and Solution Approaches

Modeling Tool Requirements
Large Scale Mixed Integer Programming Problems

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Q: Why are these problems difficult?

A: Because requirements are complex:

» Combination of continuous, binary, and integer decision variables
» Often multi-objective (weighted objectives or goal programming)
» Often with soft constraints (nice to have)
» Often infeasible unless most soft constraints are relaxed
» Lots of unknowns
Optimization Problem Structure

- Decision variables (up to 100s of millions)
  a.k.a. Columns

- Constraints (up to 10s of millions)
  a.k.a. Rows
Types of large MIP problems:
No discernible shape

Problems with tightly coupled rows and columns

Signature: Every decision almost directly impacts every other decision

Examples: Multi-party Exchanges, Scheduling, TSP

The bad news: Can be hard to solve, even for medium-size problems

The good news: Easy to model; Advanced solver tuning can pay off

Your best bet: Get the best solver and tune it; Identify patterns and reformulate through substitution/decomposition if possible
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Types of large MIP problems: Columns >> Rows

An extremely large number of columns

Signature: More variables than any computer’s memory could fit

Examples: Crew pairing, sports scheduling, vehicle routing

The bad news: Implementations may have to sacrifice optimality

The good news: Parallel implementations work well in practice

Your best bet: Column generation (master problem and iterative sub-problems)
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Column generation in Xpress-Mosel

while(true) do
  minimize(XPRS_LIN, MinRolls)  ! Solve the LP
  savebasis(bas)          ! Save the current basis
  objval:= getobjval       ! Get the objective value
  ! Get the solution values
  forall(j in 1..npatt) soluse(j):=getsol(use(j))
  forall(i in WIDTHS) dualdem(i):=getdual(Dem(i))
  ! Solve a knapsack problem
  zbest:= knapsack(dualdem, WIDTH, MAXWIDTH, xbest) - 1.0
  write("Pass ", npass, ": ")
  if zbest = 0 then          ! Same as zbest <= EPS
    writeln("no profitable column found.\n")
    break
  else
    show_new_pat(zbest, xbest)  ! Print the new pattern
    npatt+=1
    create(use(npatt))         ! Create a new var. for this pattern
    use(npatt) is_integer
    MinRolls+= use(npatt)       ! Add new var. to the objective
    dw:=0
    forall(i in WIDTHS)
      if xbest(i) > 0 then
        Dem(i):= xbest(i)*use(npatt)  ! Add new var. to demand constr.s
        dw:= maxlist(dw, ceil(DEMAND(i)/xbest(i)) )
      end-if
    end-if
    use(npatt) <= dw           ! Set upper bound on the new var.
    loadprob(MinRolls)         ! Reload the problem
    loadbasis(bas)             ! Load the saved basis
    npass+=1
  end-if
end-do
Column generation in Xpress-Mosel

************************************************************************
!  Solve the integer knapsack problem
!  \( z = \max\{cx : ax \leq b, \ x \in \mathbb{Z}^N\} \)
!  with \( b=\text{MAXWIDTH} \), \( N=\text{NWIDTHS} \)
!
! We reset the knapsack constraints to 0 at the end of this function
! so that they do not unnecessarily increase the size of the main
! problem. The same is true in the other sense: hiding the demand
! constraints while solving the knapsack problem makes life easier
! for the optimizer, but is not essential for getting the correct
! solution.
************************************************************************

function knapsack(C:array(range) of real,
    A:array(range) of real,
    B:real,
    xbest:array(range) of integer):real

declarations
    x: array(WIDTHS) of mpvar ! Knapsack variables
end-declarations
with mpproblem do
    ! Create a local subproblem
    ! Define the knapsack problem
    forall(j in WIDTHS) x(j) is_integer
    sum(j in WIDTHS) A(j)*x(j) <= B
    maximize(sum(j in WIDTHS) C(j)*x(j))
    returned:=getobjval
    forall(j in WIDTHS) xbest(j):=round(getsol(x(j)))

end-do

end-function
Types of large MIP problems: Master/Subproblems

Problems consisting of independent sub-problems connected by global constraints

Signature: Different independent business units with shared resources
Examples: Multi-period production planning, hierarchical packing
The bad news: Difficulty depends on the number of global constraints
The good news: Parallel implementations work well in practice
Your best bet: Dantzig-Wolfe decomposition, Hybrid algorithms, heuristics
! Process the proposal generated by a subproblem
procedure process_sub_result
declarations
  f: integer           ! Factory index
  ! Solution values of the proposal:
  sol_make: array(PRODS,TIME) of real  ! Amount of products made
  sol_sell: array(PRODS,TIME) of real  ! Amount of product sold
  sol_buy: array(RAW,TIME) of real     ! Amount of raw mat. bought
  sol_pstock: array(PRODS,1..NT+1) of real ! Product stock levels
  sol_rstock: array(RAW,1..NT+1) of real ! Raw mat. stock levels
  pc: real             ! Cost of the proposal
end-declarations

! Read proposal data from memory
initializations from "mempipe:noindex,sol"
  f
  sol_make sol_sell sol_buy sol_pstock sol_rstock
  pc
end-initializations

! Add the new proposal to the master problem
nPROP(f)+=1
create(weight(f,nPROP(f)))
forall(p in PRODS,t in TIME) do
  Prop_make(p,f,t,nPROP(f)):= sol_make(p,t)
  Prop_sell(p,f,t,nPROP(f)):= sol_sell(p,t)
end-do
forall(r in RAW,t in TIME) Prop_buy(r,f,t,nPROP(f)):= sol_buy(r,t)
forall(p in PRODS,t in 1..NT+1) Prop_pstock(p,f,t,nPROP(f)):= sol_pstock(p,t)
forall(r in RAW,t in 1..NT+1) Prop_rstock(r,f,t,nPROP(f)):= sol_rstock(r,t)
Prop_cost(f,nPROP(f)):= pc
writeln("Sol. for factory ", f, ":\n make: ", sol_make, ":\n sell: ", sol_sell, ":\n buy: ", sol_buy, ":\n pstock: ", sol_pstock,
       ":\n rstock: ", sol_rstock)
end-procedure
! (Re)solve the master problem
procedure solve_master(phase: integer)
forall(f in FACT)
    Convex(f) := sum (k in 1..nPROP(f)) weight(f,k) = 1
if phase=1 then
forall(p in PRODS, t in TIME)
    MxSell(p,t) :=
        sum(f in FACT, k in 1..nPROP(f)) Prop_sell(p,f,t,k)*weight(f,k) - excessS <= MXSELL(p,t)
minimize(excessS)
else
forall(p in PRODS, t in TIME)
    MxSell(p,t) :=
        sum(f in FACT, k in 1..nPROP(f)) Prop_sell(p,f,t,k)*weight(f,k) <= MXSELL(p,t)
maximize(sum(f in FACT, k in 1..nPROP(f)) Prop_cost(f,k) * weight(f,k))
end-if
writeln("Master problem objective: ", getobjval)
write("  Weights:")
forall(f in FACT, k in 1..nPROP(f)) write(" ", getsol(weight(f,k)))
writeln
end-procedure
! (Re)solve this model until it is stopped by event "PHASE_3"

repeat
wait
ev := getnextevent
Phase := getclass(ev)
if Phase = PHASE_3 then  ! Stop the execution of this model
  break
end-if
Price_convex := getvalue(ev)  ! Get new pricing data
if Phase <> PHASE_0 then
  initializations from "raw:noindex"
  Price_sell as "shm:Price_sell"
end-initializations
end-if

! (Re)solve this model
if Phase = PHASE_0 then
  maximize(Profit)
elif Phase = PHASE_1 then
  maximize(sum(p in PRODS, t in TIME) Price_sell(p,t) * sell(p,t) + Price_convex)
else  ! PHASE 2
  maximize(
    Profit - sum(p in PRODS, t in TIME) Price_sell(p,t) * sell(p,t) - 
    Price_convex)
end-if
writeln("Factory ", Factory, " - Obj: ", getobjval, 
  " Profit: ", getsol(Profit), " Price_sell: ",
  getsol(sum(p in PRODS, t in TIME) Price_sell(p,t) * sell(p,t) ),
  " Price_convex: ", Price_convex)
fflush
if getobjval > TOL then  ! Solution found: send values to master
  process_solution
elif getobjval <= TOL then  ! Problem is infeasible (Phase 0/1) or
  send(EVENT_FAILED,0)
else  ! no improved solution found (Phase 2)
  send(EVENT_READY,0)
end-if
until false
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The problem naturally decomposes into a macro problem (overall layout) and many micro problems (individual shelf layout)
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Production planning at a US food company

**Given**
- Demand and
- Available capacity (e.g. number of production lines and corresponding capacities)

**Provide**
- *optimized* production schedule at the product level for a given period of time

**Objectives**
- Meet demand as closely as possible
- Minimize setup cost of production lines when switching between two different products
- Minimize cost of labor
- Minimize inventory
- Minimize staleness
The planning period is split up into chunks

Cross-over intervals are added

Each time chunk is solved separately
Combining partial solutions

» … and the solutions are glued together

» If several solutions for each time chunk are available these solutions can be combined to different solutions for the complete problem

» A heuristic tries to improve the solutions locally

see J.D. Kelly, Chronological Decomposition Heuristic for Scheduling: A Divide & Conquer Method
A one week plan

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# A five week plan

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|----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Pktline 1 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 2 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 3 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 4 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 5 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 6 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 7 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 8 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 9 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 10 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 11 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 12 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 13 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 14 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| Pktline 15 |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
Types of large MIP problems: Multi-stage stochastic

Problems consisting of independent sub-problems connected by common columns

Signature: Stage 1 decisions in multi-period discrete stochastic problems

Examples: Problems with diverging future expectation (energy, finance)

The bad news: Combinatorial explosion when large number of stages

The good news: Decisions can be re-optimized when Stage 2 becomes Stage 1

Your best bet: Benders decomposition, simulation
Types of large MIP problems: Large assignment

Assignment problems applied to very large quasi-homogeneous populations, with continuous global constraints

Signature: Billions/Trillions of simple assignment decisions

Examples: Marketing problems, Credit line assignment, Collections

The bad news: May have to trade some optimality for speed

The good news: Can decompose aggressively

Your best bet: Random partitioning, but keep an eye on solution quality; use Dantzig-Wolfe or alternative if optimality is crucial
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Types of large MIP problems: Aggregated

Independent problems – when decentralized, individual planning makes sense

Signature: Heterogeneous production units with independent budgets
Examples: Routing to specialized repair facilities; production without inventory
The bad news: Almost none
The good news: A few lines of model code is all that’s required
Your best bet: Divide and conquer; Parallelize
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Summary of Problem Types and Solution Approaches

» Every real-life problem has some exploit able structure

» Known algorithms can be implemented to attack difficult problems

» The MIP solver will do the heavy lifting in finding solutions

» The MIP solver provides proof of optimality or near optimality
Requirements for Very Large Scale Optimization

» Technology enablers:
  » Hardware
    » 64 bit systems
    » Multi-CPU/Multi-core systems
  » Software
    » The fastest LP/MIP solver that:
      » Implements all state of the art cutting, branching and heuristic techniques
      » Allows search customization and fine tuning
      » Provides automatic tuning tools
    » A powerful modeling system that offers:
      » Programming language features (imperative, modular, fast data processing)
      » Built-in decomposition capabilities
      » Built-in parallelism
      » Fast in-memory data transfer
      » Advanced debugging and logging features
      » Rapid visualization for detecting patterns in the problem structure
    » A development environment with:
      » Detailed solver tracking
      » Wizards and a large variety of examples
FICO Xpress Optimization Suite
Key features for large-scale optimization

» Xpress-Optimizer efficiently solves LP and MIP problems with billions of nonzero coefficients
  » Built-in parallelism, advanced cutting, heuristics

» Xpress-Mosel is a high-level modeling language combined with the standard functionality of a programming language
  » Implementation of models and solution algorithms in a single environment; built-in parallelism, model separation, modularity

» Xpress-IVE and Xpress-Tuner support model building
  » Enhanced productivity through the most advanced debugging, profiling, tuning, and execution tracking features on the market

“My team is more productive because they have more access to these kinds of tools. It was a hit in my group when we signed the agreement with FICO.”

-- Armando Silva, managing director of operations research, American Airlines
THANK YOU

Q & A

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